CS 70 Fall 2018 — Discrete Mathematics and Probability Theory — Alistair Sinclair and Yun Song — DIS 3A

## 1 Touring Hypercube

In the lecture, you have seen that if $G$ is a hypercube of dimension $n$, then

- The vertices of $G$ are the binary strings of length $n$.
- $u$ and $v$ are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that:

- Each vertex appears exactly once in the sequence.
- Each pair of consecutive vertices is connected by an edge.
- $v_0$ and $v_k$ are connected by an edge.

(a) Show that a hypercube has an Eulerian tour if and only if $n$ is even.

(b) Show that every hypercube has a Hamiltonian tour.

## 2 Trees

Recall that a *tree* is a connected acyclic graph (graph without cycles). In the note, we presented a few other definitions of a tree, and in this problem, we will prove two fundamental properties of a tree, and derive two definitions of a tree we learned from the note based on these properties. Let's start with the properties:

(a) Prove that any pair of vertices in a tree are connected by exactly one (simple) path.

(b) Prove that adding any edge (not already in the graph) between two vertices of a tree creates a simple cycle.

Now you will show that if a graph satisfies this property then it must be a tree:

(c) Prove that if the graph has no simple cycles and has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

# 3 Planarity

Consider graphs with the property $T$: For every three distinct vertices $v_1, v_2, v_3$ of graph $G$, there are at least two edges among them. Prove that if $G$ is a graph on $\geq 7$ vertices, and $G$ has property $T$, then $G$ is nonplanar.