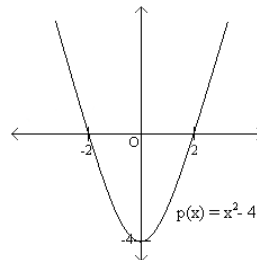


Polynomials

Polynomials constitute a rich class of functions which are both easy to describe and widely applicable in topics ranging from Fourier analysis, cryptography and communication to control and computational geometry. In this note, we will discuss properties of polynomials which make them so useful. We will then describe how to take advantage of these properties to develop a secret sharing scheme.

Recall from your high school math that a *polynomial* in a single variable is of the form $p(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$. Here the *variable* x and the *coefficients* a_i are usually real numbers. For example, $p(x) = 5x^3 + 2x + 1$, is a polynomial of *degree* $d = 3$. Its coefficients are $a_3 = 5$, $a_2 = 0$, $a_1 = 2$, and $a_0 = 1$. Polynomials have some remarkably simple, elegant and powerful properties, which we will explore in this note.

First, a definition: we say that a is a *root* of the polynomial $p(x)$ if $p(a) = 0$. For example, the degree 2 polynomial $p(x) = x^2 - 4$ has two roots, namely 2 and -2 , since $p(2) = p(-2) = 0$. If we plot the polynomial $p(x)$ in the x - y plane, then the roots of the polynomial are just the places where the curve crosses the x axis:

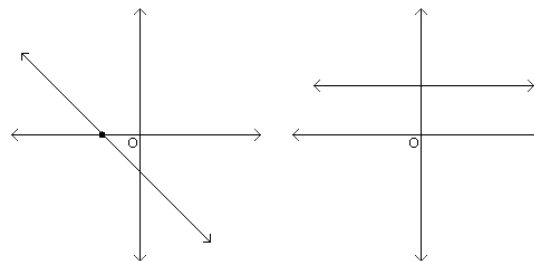


We now state two fundamental properties of polynomials that we will prove in due course.

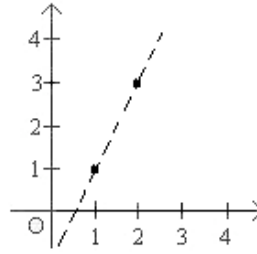
Property 1: A non-zero polynomial of degree d has at most d roots.

Property 2: Given $d + 1$ pairs $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$, with all the x_i distinct, there is a unique polynomial $p(x)$ of degree (at most) d such that $p(x_i) = y_i$ for $1 \leq i \leq d + 1$.

Let us consider what these two properties say in the case that $d = 1$. A graph of a linear (degree 1) polynomial $y = a_1 x + a_0$ is a line. Property 1 says that if a line is not the x -axis (i.e. if the polynomial is not $y = 0$), then it can intersect the x -axis in at most one point.



Property 2 says that two points uniquely determine a line.



Polynomial Interpolation

Property 2 says that two points uniquely determine a degree 1 polynomial (a line), three points uniquely determine a degree 2 polynomial, four points uniquely determine a degree 3 polynomial, and so on. Given $d + 1$ pairs $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$, how do we determine the polynomial $p(x) = a_d x^d + \dots + a_1 x + a_0$ such that $p(x_i) = y_i$ for $i = 1$ to $d + 1$? We will give an efficient algorithms for reconstructing the coefficients a_0, \dots, a_d , and therefore the polynomial $p(x)$.

The method is called *Lagrange interpolation*: Let us start by solving an easier problem. Suppose that we are told that $y_1 = 1$ and $y_j = 0$ for $2 \leq j \leq d + 1$. Now can we reconstruct $p(x)$? Yes, this is easy! Consider $q(x) = (x - x_2)(x - x_3) \cdots (x - x_{d+1})$. This is a polynomial of degree d (the x_i 's are constants, and x appears d times). Also, we clearly have $q(x_j) = 0$ for $2 \leq j \leq d + 1$. But what is $q(x_1)$? Well, $q(x_1) = (x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_{d+1})$, which is some constant not equal to 0 (since the x_i are all distinct). Thus if we let $p(x) = q(x)/q(x_1)$ (dividing is ok since $q(x_1) \neq 0$), we have the polynomial we are looking for. For example, suppose you were given the pairs $(1, 1)$, $(2, 0)$, and $(3, 0)$. Then we can construct the degree $d = 2$ polynomial $p(x)$ by letting $q(x) = (x - 2)(x - 3) = x^2 - 5x + 6$, and $q(x_1) = q(1) = 2$. Thus, we can now construct $p(x) = q(x)/q(x_1) = (x^2 - 5x + 6)/2$.

Of course, the problem is no harder if we single out some arbitrary index i instead of 1: i.e. $y_i = 1$ and $y_j = 0$ for $j \neq i$. Let us introduce some notation: let us denote by $\Delta_i(x)$ the degree d polynomial that goes through these $d + 1$ points. Then $\Delta_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$.

Let us now return to the original problem. Given $d + 1$ pairs $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$, we first construct the $d + 1$ polynomials $\Delta_1(x), \dots, \Delta_{d+1}(x)$ as described above. Now we can write $p(x) = \sum_{i=1}^{d+1} y_i \Delta_i(x)$. Why does this work? First notice that $p(x)$ is a polynomial of degree d as required, since it is the sum of polynomials of degree d . And when it is evaluated at x_i , d of the $d + 1$ terms in the sum evaluate to 0 and the i -th term evaluates to y_i times 1, as required.

As an example, suppose we want to find the degree-2 polynomial $p(x)$ that passes through the three points $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (2, 2)$ and $(x_3, y_3) = (3, 4)$. The three polynomials Δ_i are as follows:

$$\begin{aligned} \Delta_1(x) &= \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2} = \frac{1}{2}x^2 - \frac{5}{2}x + 3; \\ \Delta_2(x) &= \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{(x-1)(x-3)}{-1} = -x^2 + 4x - 3; \\ \Delta_3(x) &= \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{(x-1)(x-2)}{2} = \frac{1}{2}x^2 - \frac{3}{2}x + 1. \end{aligned}$$

The polynomial $p(x)$ is therefore given by

$$p(x) = 1 \cdot \Delta_1(x) + 2 \cdot \Delta_2(x) + 4 \cdot \Delta_3(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 1.$$

You should verify that this polynomial does indeed pass through the above three points.

Proof of Property 2

We are now in a position to prove Property 2 stated earlier, namely:

Property 2: Given $d + 1$ pairs $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$, with all the x_i distinct, there is a unique polynomial $p(x)$ of degree (at most) d such that $p(x_i) = y_i$ for $1 \leq i \leq d + 1$.

We have shown above how to find a polynomial $p(x)$ such that $p(x_i) = y_i$ for $d + 1$ pairs $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$. This proves part of Property 2 (the existence of the polynomial). How do we prove the second part, that the polynomial is unique? Suppose for contradiction that there is another polynomial $q(x)$ such that $q(x_i) = y_i$ for all $d + 1$ pairs above. Now consider the polynomial $r(x) = p(x) - q(x)$. Since we are assuming that $q(x)$ and $p(x)$ are different polynomials, $r(x)$ must be a non-zero polynomial of degree at most d . Therefore, Property 1 (to be proved below) implies it can have at most d roots. But on the other hand $r(x_i) = p(x_i) - q(x_i) = 0$ on $d + 1$ distinct points. Contradiction. Therefore, $p(x)$ is the unique polynomial that satisfies the $d + 1$ conditions.

Polynomial Division

Let's take a short digression to discuss polynomial division, which will be useful in the proof of Property 1. If we have a polynomial $p(x)$ of degree d , we can divide by a polynomial $q(x)$ of degree $\leq d$ by using long division. The result will be:

$$p(x) = q'(x)q(x) + r(x)$$

where $q'(x)$ is the quotient and $r(x)$ is the remainder. The degree of $r(x)$ must be smaller than the degree of $p(x)$.

Example. We wish to divide $p(x) = x^3 + x^2 - 1$ by $q(x) = x - 1$:

- First we subtract a factor $x^2(x - 1)$ to write: $p(x) = x^2(x - 1) + (2x^2 - 1)$.
- Then we subtract a factor $2x(x - 1)$ to write the remainder as $2x^2 - 1 = 2x(x - 1) + (2x - 1)$.
- Then we subtract a factor $2(x - 1)$ to write the remainder as $2x - 1 = 2(x - 1) + 1$.
- Finally, putting the above three lines together, we get that $p(x) = (x^2 + 2x + 2)(x - 1) + 1$.

Therefore, the quotient is $q'(x) = x^2 + 2x + 2$ and the remainder is $r(x) = 1$.

Proof of Property 1

Now let us turn to Property 1: a non-zero polynomial of degree d has at most d roots. The idea of the proof is as follows. We will prove the following claims:

Claim 1 If a is a root of a polynomial $p(x)$ with degree d , then $p(x) = (x - a)q(x)$ for a polynomial $q(x)$ with degree $d - 1$.

Claim 2 A polynomial $p(x)$ of degree d with distinct roots a_1, \dots, a_d can be written as $p(x) = c(x - a_1) \cdots (x - a_d)$, where c is a real number.

Note that Claim 2 immediately implies Property 1: we just need to show that $a \neq a_i$ for $i = 1, \dots, d$ cannot be a root of $p(x)$. But this follows from Claim 2, since $p(a) = c(a - a_1) \cdots (a - a_d) \neq 0$.

Proof of Claim 1

Dividing $p(x)$ by $(x - a)$ gives $p(x) = (x - a)q(x) + r(x)$, where $q(x)$ is the quotient and $r(x)$ is the remainder. The degree of $r(x)$ is necessarily smaller than the degree of the divisor $(x - a)$. Therefore $r(x)$ must have degree 0 and therefore is some constant c . But now substituting $x = a$, we get $p(a) = c$. But since a is a root, $p(a) = 0$. Thus $c = 0$ and therefore $p(x) = (x - a)q(x)$.

Claim 1 implies Claim 2

Proof by induction on d .

- **Base Case:** We must show that a polynomial $p(x)$ of degree 1 with root a_1 can be written as $p(x) = c(x - a_1)$. By Claim 1, we know that $p(x) = (x - a_1)q(x)$, where $q(x)$ has degree 0 and is therefore a constant.
- **Inductive Hypothesis:** A polynomial of degree $d - 1$ with distinct roots a_1, \dots, a_{d-1} can be written as $p(x) = c(x - a_1) \cdots (x - a_{d-1})$.
- **Inductive Step:** Let $p(x)$ be a polynomial of degree d with distinct roots a_1, \dots, a_d . By Claim 1, $p(x) = (x - a_d)q(x)$ for some polynomial $q(x)$ of degree $d - 1$. Since $0 = p(a_i) = (a_i - a_d)q(a_i)$ for all $i \neq d$ and $a_i - a_d \neq 0$ in this case, $q(a_i)$ must be equal to 0. Then $q(x)$ is a polynomial of degree $d - 1$ with distinct roots a_1, \dots, a_{d-1} . We can now apply the inductive assumption to $q(x)$ to write $q(x) = c(x - a_1) \cdots (x - a_{d-1})$. Substituting in $p(x) = (x - a_d)q(x)$, we finally obtain that $p(x) = c(x - a_1) \cdots (x - a_d)$.

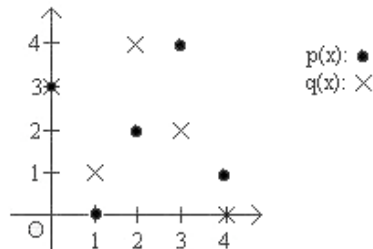
Finite Fields

Both Property 1 and Property 2 also hold when the values of the coefficients and the variable x are chosen from the complex numbers instead of the real numbers or even the rational numbers. They do not hold if the values are restricted to being natural numbers or integers. Let us try to understand this a little more closely. The only properties of numbers that we used in polynomial interpolation and in the proof of Property 1 are that we can add, subtract, multiply and divide any pair of numbers as long as we are not dividing by 0. We cannot subtract two natural numbers and guarantee that the result is a natural number. And dividing two integers does not usually result in an integer.

But if we work with numbers modulo a prime m , then we can add, subtract, multiply and divide (by any non-zero number modulo m). To check this, recall that x has an inverse mod m if $\gcd(m, x) = 1$, so if m is prime *all* the numbers $\{1, \dots, m - 1\}$ have an inverse mod m . So both Property 1 and Property 2 hold if the coefficients and the variable x are restricted to take on values modulo m . This remarkable fact—that these properties hold even when we restrict ourselves to a *finite* set of values—is the key to several applications that we will presently see.

Let us consider an example of degree $d = 1$ polynomials modulo 5. Let $p(x) = 2x + 3 \pmod{5}$. The roots of this polynomial are all values x such that $2x + 3 \equiv 0 \pmod{5}$ holds. Solving for x , we get that $2x = -3 \equiv 2 \pmod{5}$, and thus $x = 1 \pmod{5}$. Note that this is consistent with Property 1 since we got only one root of a degree-1 polynomial.

Now consider the polynomials $p(x) = 2x + 3 \pmod{5}$ and $q(x) = 3x - 2 \pmod{5}$. We can plot the values y of each polynomial as a function of x in the x - y plane. Since we are working modulo 5, there are only 5 possible choices for x , and only 5 possible choices for y :



Notice that these two “lines” intersect in exactly one point, even though the picture looks nothing at all like lines in the Euclidean plane! Looking at these graphs it might seem remarkable that both Property 1 and Property 2 hold when we work modulo m for any prime number m . But as we stated above, all that was required for the proofs of Property 1 and 2 was the ability to add, subtract, multiply and divide any pair of numbers (as long as we are not dividing by 0), and they hold whenever we work modulo a prime m .

When we work with numbers modulo a prime m , we say that we are working over a finite field, denoted by F_m or $GF(m)$ (for Galois Field). In order for a set to be called a field, it must satisfy certain axioms which are the building blocks that allow for these properties and others to hold. If you would like to learn more about fields and the axioms they satisfy, you can visit Wikipedia’s site and read the article on fields: http://en.wikipedia.org/wiki/Field_%28mathematics%29. While you are there, you can also read the article on Galois Fields and learn more about some of their applications and elegant properties which will not be covered in this lecture: http://en.wikipedia.org/wiki/Galois_field.

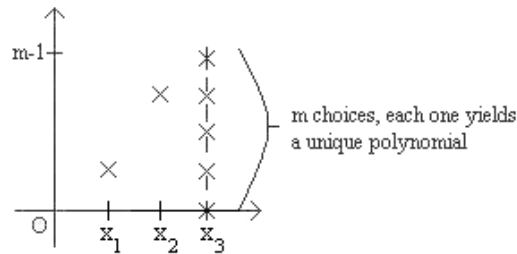
Counting

How many polynomials of degree (at most) 2 are there modulo m ? This is easy: there are 3 coefficients, each of which can take on one of m values for a total of m^3 . Writing $p(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_0$ by specifying its $d + 1$ coefficients a_i is known as the *coefficient representation* of $p(x)$. Is there any other way to specify $p(x)$?

Sure, there is! Our polynomial of degree (at most) 2 is uniquely specified by its values at any three points, say $x = 0, 1, 2$. Once again, the polynomial can take any one of m values at each of these three points, for a total of m^3 possibilities. In general, we can specify a degree d polynomial $p(x)$ by specifying its values at $d + 1$ points, say $0, 1, \dots, d$. These $d + 1$ values, (y_0, y_1, \dots, y_d) , are called the *value representation* of $p(x)$. The coefficient representation can be converted to the value representation by evaluating the polynomial at $0, 1, \dots, d$. And, as we’ve seen, Lagrange interpolation can be used to convert the value representation to the coefficient representation.

So if we are given three pairs $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ then there is a unique polynomial of degree 2 such that $p(x_i) = y_i$. But now, suppose we were only given two pairs $(x_1, y_1), (x_2, y_2)$; how many distinct degree-2 polynomials are there that go through these two points? Notice that there are exactly m choices for y_3 , and for each choice there is a unique (and distinct) polynomial of degree two that goes through the three points

$(x_1, y_1), (x_2, y_2), (x_3, y_3)$. It follows that there are exactly m polynomials of degree at most 2 that go through two points $(x_1, y_1), (x_2, y_2)$, as shown below:



What if you were only given one point (x_1, y_1) ? Well, there are m^2 choices for y_2 and y_3 , yielding m^2 polynomials of degree at most 2 that go through the point given. A pattern begins to emerge, as is summarized in the following table:

Polynomials of degree $\leq d$ over F_m	
# of points	# of polynomials
$d + 1$	1
d	m
$d - 1$	m^2
\vdots	\vdots
$d - k$	m^{k+1}
\vdots	\vdots
0	m^{d+1}

Note that the reason that we can count the number of polynomials in this setting is because we are working over a finite field. If we were working over an infinite field such as the reals, there would be infinitely many polynomials of degree d that can go through d points! Think of a line, which has degree one. If you were just given one point, there would be infinitely many possibilities for the second point, each of which uniquely defines a line.

Secret Sharing

In the late 1950's and into the 1960's, during the Cold War, President Dwight D. Eisenhower approved instructions and authorized top commanding officers for the use of nuclear weapons under very urgent emergency conditions. Such measures were set up in order to defend the United States in case of an attack in which there was not enough time to confer with the President and decide on an appropriate response. This would allow for a rapid response in case of a Soviet attack on U.S. soil. This is a perfect situation in which a secret sharing scheme could be used to ensure that a certain number of officials must come together in order to successfully launch a nuclear strike, so that for example no single person has the power and control over such a devastating and destructive weapon. Suppose the U.S. government finally decides that a nuclear strike can be initiated only if at least $k > 1$ major officials agree to it. We want to devise a scheme such that both of the following properties hold:

1. Any group of k of these officials can pool their information to figure out the launch code and initiate the strike.

- No group of $k - 1$ or fewer have *any* information about the launch code, even if they pool their knowledge. For example, they should not learn whether the secret is odd or even, a prime number, divisible by some number a , or the secret's least significant bit.

How can we accomplish this?

Suppose that there are n officials indexed from 1 to n and the launch code is some natural number s . Let q be a prime number larger than n and s . We will work over $GF(q)$ from now on.

Now pick a random polynomial $P(x)$ of degree $k - 1$ such that $P(0) = s$ and give $P(1)$ to the first official, $P(2)$ to the second, \dots , $P(n)$ to the n^{th} . Then we have:

- Any k officials, having the values of the polynomial at k points, can use Lagrange interpolation to find P , and once they know what P is, they can compute $P(0) = s$ to learn the secret. Another way to say this is that any k officials have between them a value representation of the polynomial, which they can convert to the coefficient representation, which allows them to evaluate $P(0) = s$.
- Any group of $k - 1$ (or fewer) officials has no information about s . To see this, observe that they know only $k - 1$ points through which $P(x)$, an unknown polynomial of degree $k - 1$, passes. They wish to reconstruct $P(0) = s$. But by our discussion in the previous section, for each possible value $P(0) = b$, there is a unique polynomial of degree $k - 1$ that passes through the $k - 1$ points that the $k - 1$ officials have as well as through $(0, b)$. Hence the secret could be *any* of the q possible values $\{0, 1, \dots, q - 1\}$, so the officials have—in a very precise sense—no information about s . Another way of saying this is that the information of the officials is consistent with q different value representations, one for each possible value of the secret, and thus the officials have no information¹ about s .

Example. Suppose you are in charge of setting up a secret sharing scheme, with secret $s = 1$, where you want to distribute $n = 5$ shares to 5 people such that any $k = 3$ or more people can figure out the secret, but two or fewer cannot. Let's say we are working over $GF(7)$ (note that $7 > s$ and $7 > n$) and you randomly choose the following polynomial of degree $k - 1 = 2$: $P(x) = 3x^2 + 5x + 1$ (here, $P(0) = 1 = s$, the secret). So you know everything there is to know about the secret and the polynomial, but what about the people that receive the shares? Well, the shares handed out are $P(1) = 2$ to the first official, $P(2) = 2$ to the second, $P(3) = 1$ to the third, $P(4) = 6$ to the fourth, and $P(5) = 3$ to the fifth official. Let's say that officials 3, 4, and 5 get together (we expect them to be able to recover the secret). Using Lagrange interpolation, they compute the following delta functions:

$$\begin{aligned}\Delta_3(x) &= \frac{(x-4)(x-5)}{(3-4)(3-5)} = \frac{(x-4)(x-5)}{2} = 4(x-4)(x-5); \\ \Delta_4(x) &= \frac{(x-3)(x-5)}{(4-3)(4-5)} = \frac{(x-3)(x-5)}{-1} = 6(x-3)(x-5); \\ \Delta_5(x) &= \frac{(x-3)(x-4)}{(5-3)(5-4)} = \frac{(x-3)(x-4)}{2} = 4(x-3)(x-4).\end{aligned}$$

They then compute the polynomial over $GF(7)$: $P(x) = (1)\Delta_3(x) + (6)\Delta_4(x) + (3)\Delta_5(x) = 3x^2 + 5x + 1$ (verify the computation!). Now they simply compute $P(0)$ and discover that the secret is 1.

¹Note that this is one reason we choose to work over finite fields rather than, say, over the real numbers, where the basic secret-sharing scheme would still work. Because there are only finitely many values in our field, we can quantify precisely how many remaining possibilities there are for the value of the secret, and show that this is the same as if the officials had no information at all.

Let's see what happens if two officials try to get together, say persons 1 and 5. They both know that the polynomial looks like $P(x) = a_2x^2 + a_1x + s$. They also know the following equations:

$$P(1) = a_2 + a_1 + s = 2$$

$$P(5) = 4a_2 + 5a_1 + s = 3$$

But that is all they have—two equations with three unknowns—and thus they cannot find out the secret. This is the case no matter which two officials get together. Notice that since we are working over $GF(7)$, the two people could have guessed the secret ($0 \leq s \leq 6$) and constructed a unique degree 2 polynomial (by Property 2). But the two people combined have the same chance of guessing what the secret is as they do individually. This is important, as it implies that two people have no more information about the secret than one person does.