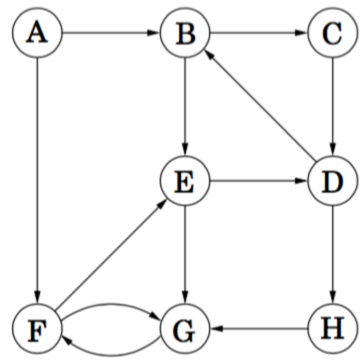
CS 70 Discrete Mathematics and Probability Theory
Fall 2018 Alistair Sinclair and Yun Song DIS 2B

1 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $(B,C),(C,D),(D,B)$
ii. $(F,G),(G,F)$
iii. $(A,B),(B,C),(C,D),(D,B)$
iv. $(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)$

(e) Which of the following are walks in G?

i. (E,G)

ii. $(E,G),(G,F)$

iii. $(F,G),(G,F)$

iv. $(A,B),(B,C),(C,D),(H,G)$

v. $(E,G),(G,F),(F,G),(G,C)$

vi. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

(f) Which of the following are tours in G?

i. (E,G)

ii. $(E,G),(G,F)$

iii. $(F,G),(G,F)$

iv. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

2 Odd Degree Vertices

Claim: Let $G=(V,E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

(ii) Induction on $m=|E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)

(iv) Well-ordering principle (*Hint*: Try rephrasing one of the induction proofs.)

3 Touring Hypercube

In the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \dots, v_k$ such that:

- Each vertex appears exactly once in the sequence.
- Each pair of consecutive vertices is connected by an edge.
- v_0 and v_k are connected by an edge.

(a) Show that a hypercube has an Eulerian tour if and only if n is even.

(b) Show that every hypercube has a Hamiltonian tour.