1   More Countability

Given:

- A is a countable, non-empty set. For all $i \in A$, S_i is an uncountable set.

- B is an uncountable set. For all $i \in B$, Q_i is a countable set.

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $A \cap B$

(b) $A \cup B$

(c) $\bigcup_{i \in A} S_i$

(d) $\bigcap_{i \in A} S_i$

(e) $\bigcup_{i \in B} Q_i$

(f) $\bigcap_{i \in B} Q_i$

2 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

(a) You want to determine whether a program P on input x prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

(b) You want to determine whether a program P prints "Hello World!" before running the kth line in the program. Is there a computer program that can perform this task? Justify your answer.

(c) You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task? Justify your answer.

3 Computability

Decide whether the following statements are true or false. Please justify your answers.

(a) The problem of determining whether a program halts in time 2^{n^2} on an input of size n is undecidable.

(b) There is no computer program `Line` which takes a program P, an input x, and a line number L, and determines whether the Lth line of code is executed when the program P is run on the input x.