CS 70 Fall 2018 — Discrete Mathematics and Probability Theory — Alistair Sinclair and Yun Song — DIS 07A

1 Counting Mappings

(a) Let $X = \{1,2,\ldots,n\}$ and $Y = \{1,2,\ldots,m\}$. How many distinct functions f are there from X to Y?

(b) How many of these functions f are injective?

(c) For functions f as in part (a), consider the (ordered) lists $\left(|f^{-1}(1)|,|f^{-1}(2)|,\ldots,|f^{-1}(m)|\right)$. How many distinct such lists are there? How many are there if we only consider surjective functions?

2 Counting on Graphs

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself.

(b) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) How many distinct cycles are there in a complete graph with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

3 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$.

(b) $\binom{n}{i} = \binom{n}{n-i}$.

(c) $\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}$.