Due: Friday, November 16 at 10PM

Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Safeway Monopoly Cards

It's that time of the year again - Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\text{var}(X) = n^2 \left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}(X)$. *[Hint: Does this remind you of a particular problem? What is the expectation for this problem?]*

2 Geometric Distribution

Two faulty machines, M_1 and M_2, are repeatedly run synchronously in parallel (i.e., both machines execute one run, then both execute a second run, and so on). On each run, M_1 fails with probability p_1 and M_2 fails with probability p_2, all failure events being independent. Let the random variables X_1, X_2 denote the number of runs until the first failure of M_1, M_2 respectively; thus X_1, X_2 have geometric distributions with parameters p_1, p_2 respectively. Let X denote the number of runs until the first failure of *either* machine.

(a) Show that X also has a geometric distribution, with parameter $p_1 + p_2 - p_1 p_2$.

(b) Now, two technicians are hired to check on the machines every run. They decide to take turns checking on the machines every hour. What is the probability that the first technician is the first one to find a faulty machine?

3 Geometric and Poisson

Let X be geometric with parameter p, Y be Poisson with parameter λ, and $Z = \max(X, Y)$. Assume X and Y are independent. For each of the following parts, your final answers should not have summations.

(a) Compute $P(X > Y)$.

(b) Compute $P(Z \geq X)$.

(c) Compute $P(Z \leq Y)$.

4 Darts

Alvin is playing darts. His aim follows an exponential distribution; that is, the probability density that the dart is x distance from the center is $f_X(x) = \exp(-x)$. The board's radius is 4 units.

(a) What is the probability the dart will stay within the board?

(b) Say you know Alvin made it on the board. What is the probability he is within 1 unit from the center?

(c) If Alvin is within 1 unit from the center, he scores 4 points, if he is within 2 units, he scores 3, etc. In other words, Alvin scores $\lfloor 5 - x \rfloor$, where x is the distance from the center. What is Alvin's expected score after one throw?

5 Exponential Practice

(a) Let $X_1, X_2 \sim \text{Exponential}(\lambda)$ be independent, $\lambda > 0$. Calculate the density of $Y := X_1 + X_2$. [*Hint*: One way to approach this problem would be to compute the CDF of Y and then differentiate the CDF.]

(b) Let $t > 0$. What is the density of X_1, conditioned on $X_1 + X_2 = t$? [*Hint*: Once again, it may be helpful to consider the CDF $\mathbb{P}(X_1 \leq x \mid X_1 + X_2 = t)$. To tackle the conditioning part, try conditioning instead on the event $\{X_1 + X_2 \in [t, t + \varepsilon]\}$, where $\varepsilon > 0$ is small.]

6 Uniform Means

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed uniform random variables on the interval $[0, 1]$ (where n is a positive integer).

(a) Let $Y = \min\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Y)$. [*Hint*: Use the tail sum formula, which says the expected value of a nonnegative random variable is $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\,dx$. Note that we can use the tail sum formula since $Y \geq 0$.]

(b) Let $Z = \max\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Z)$. [*Hint*: Find the CDF.]