Combinatorial Proofs

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What are combinatorial proofs? In general, this class of proofs involves reasoning about two expressions logically. It is crucial that you do *not* commit the following two common mistakes:

1. Do not prove the statement with equations. This means expanding the choose statements binomially. With that said, you may feel free to convert between equivalent expressions that are more intuitive.

2. Do not simply restate the equations in English. ("We sum from $i = 0$ to $i = k$, where we pick i from all n ...")

So, what *is* a combinatorial proof? We give *interpretations* to the left and right sides of the equation and show that the two sides of the equation are really two different ways of counting the same quantity; therefore, the left and right sides must be equal!

These proofs are difficult when starting off, but given time and practice, they can be mastered. The goal is to develop a deeper intuition for what it means to choose or to permute, to add or to multiply combinations. For now, let's jump into an example to make the idea more concrete.

1 A First Example

We will prove the following identity:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Both sides of the identity are counting the number of subsets of a set of size n. For each element of the set, we can decide whether to include it in the subset or not. This yields two choices for each of the n elements, for a total of 2^n: this is the left side.

On other other hand, $\binom{n}{k}$ is the number of subsets of size k. Since we need to account for subsets of all possible sizes, we need to sum over all values of k from 0 to n: this yields the right side.

We have shown that both sides of the equation count the same quantity, so by combinatorial argument, the two sides of the equation are equal.

2 Strategies

Several of the following are broadly applicable, for all sections in probability. However, we will introduce them here, as part of a set of approaches you can use to tackle combinatorial proofs.

- Addition is OR, and multiplication is AND.
  - "Expand" coefficients. $2\binom{n}{2} = \binom{n}{2} + \binom{n}{2}$, so consider all pairs of elements from a set of size n OR all pairs from (another) n.
  - Distribute quantities. $\binom{n}{2}^{a+b} = \binom{n}{2}^a \binom{n}{2}^b$, so we consider all pairs from n, a times, AND all pairs from (another) n, b times.
- Switch between equivalent forms for combinations, to see which makes more sense.
  - Rewrite quantities as "choose 1". $n = \binom{n}{1}$, so we pick one from n items.
  - Toggle between the two quantities: $\binom{a+b}{a} = \binom{a+b}{b}$. This is because choosing a elements to be in your set is the same as choosing b elements to *not* be in your set.
- Try applying the first rule of counting as well.

- 2^n is the number of all (possibly empty) subsets of a set of size n. In other words, we consider two possibilities for all n items: $\{\textsc{Include}, \textsc{Don't Include}\}$.
- $\binom{n}{k}k! = \frac{n!}{(n-k)!}$, which is the number of *ordered* subsets of size k (from a set of size n).

- Make sure to not to use equations to prove your result.

3 More Examples

Try to do these on your own first. The key to learning how to do combinatorial proofs is to pick up on the common patterns and tricks.

1. $$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

   Solution: From n students, we elect a student council by choosing r students, and among the student council, we elect k officers. For the right side, we start by choosing the k officers, and then we choose the $r-k$ other members of the student council.

2. $$\binom{2n}{n} = 2\binom{2n-1}{n-1}$$

   Solution: This one is really tricky! The factor of 2 multiplying the right side is not easily interpretable, until we write $2\binom{2n-1}{n-1}$ as $\binom{2n-1}{n-1} + \binom{2n-1}{n-1}$. Then, observe that $\binom{2n-1}{n-1} = \binom{2n-1}{n}$ (using the identity $\binom{a+b}{a} = \binom{a+b}{b}$). Now, we are ready to present the story. $2n$ students are auditioning for n spots in the school play. The left side directly counts the number of ways to choose the n actors. On the right side, we consider a more indirect scheme: suppose the first student to audition is named Sarah. We can choose to keep Sarah, in which case we have to choose $n-1$ more actors from the remaining $2n-1$ students. If we choose to reject Sarah, then we have to choose n more actors from the remaining $2n-1$ students. Since we have to count both possibilities, we add $\binom{2n-1}{n-1} + \binom{2n-1}{n}$, which is the right side of the identity.

Remark: Once we have rewritten the identity to say $\binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n}$, then we see that this is basically Pascal's identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

3.
$$\sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} = 3^n$$

Solution: The right side is easier to interpret, so we start with that. We take a set of size n, and for each element, we decide whether the element goes in set A, set B, or set C (we are splitting the elements of the larger set among three other sets). For the left side, suppose we already know that we want i elements in A and j elements in B. Then, we have $\binom{n}{i}$ ways to choose the elements of A, and $\binom{n-i}{j}$ ways to choose the elements of B once we have already chosen the elements of A. (After we have chosen the elements of A and B, the rest of the elements go to set C.) Finally, we have to sum over all of the possible values of i and j.