Vincent's CS70 Discussion 14B Notes

April 29, 2016

1 Is there any field where Gauss doesn't show up?

We now proceed to study what is arguably the most important of all continuous distributions: the Gaussian distribution (also referred to as the normal distribution).

Definition 1. *Let $\mu, \sigma \in \mathbb{R}$. A random variable Y is said to be a Gaussian, written $Y \sim \mathcal{N}(\mu, \sigma^2)$, if Y has pdf given by*

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2}.$$

The case where $\mu = 0$ and $\sigma = 1$ is referred to as the standard normal distribution.

Our next theorem is extremely important. Informally, it states that we can spend most of our time studying the standard normal distribution (as opposed to more general Gaussians), as all other Gaussians can be obtained by scaling and shifting the standard normal. Before proceeding to prove our theorem, we'll first need a lemma.

Lemma 1. *The cdf of a standard normal random variable $X \sim \mathcal{N}(0,1)$ is*

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$$

Proof. This result follows more or less immediately from the definition of the standard normal distribution and the fact the cdf and pdf of X are related by the equation

$$F_X(x) = \int_{-\infty}^{x} f(s) ds$$

(see the last discussion note at the end of section 2 to brush up on why) □

Theorem 1. *Let X be a random variable following the standard normal distribution. That is, let $X = \mathcal{N}(0,1)$. If we set $Y = \mu + \sigma X$, then $Y = \mathcal{N}(\mu, \sigma^2)$.*

Proof. Begin by noting that since $Y = \mu + \sigma X$, we have that $X = \frac{Y-\mu}{\sigma}$ is a standard normal random variable. We aim to show that $Y \sim \mathcal{N}(\mu, \sigma^2)$ by finding its cdf and then differentiating it. Recalling that the cdf of Y is the function such that $F_Y(y) = \mathbb{P}(Y \leq y)$, we have

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\mu + \sigma X \leq y) = \mathbb{P}(X \leq \frac{y-\mu}{\sigma}) = F_X(\frac{y-\mu}{\sigma}),$$

where F_X is the cdf of X.

Now, we differentiate $F_Y(y)$ to see that

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}(F_Y(y)) \\ &= \frac{d}{dy}(F_X(\frac{y-\mu}{\sigma})) \\ &= \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}e^{-(y-\mu)^2/2\sigma^2} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-\mu)^2/2\sigma^2}, \end{aligned}$$

where the third line follows from the (possibly long forgotten) chain rule from calculus. □

As a first example of how the theorem above comes in handy, we have the following result.

Corollary 1. *Let* $Y \sim \mathcal{N}(\mu, \sigma^2)$. *Then,*

$$E[Y] = \mu \text{ and } Var(Y) = \sigma^2.$$

Proof. We show the result for $X \sim \mathcal{N}(0,1)$, and from there our result follows from properties of expectation and variance, along with the fact that we can write $Y = \mu + \sigma X$. First, we see that

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}}e^{-x^2/2}dx = 0$$

as x is an odd function, e^{-x^2} is an even function, so their product is an odd function and we are integrating with symmetric bounds.

To see that $Var(X) = 1$, we examine that as we computed $E[X] = 0$ above.

$$Var(X) = E[X^2] - (E[X])^2 = E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}}e^{-x^2/2}dx.$$

From here, to show that $\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}}e^{-x^2/2}dx = 1$ becomes an exercise in elementary calculus that we leave to the especially bored reader to verify. □

2 The Central Limit Theorem

We unfortunately won't have the time to prove it, but we now proceed to state the Central Limit Theorem, one of the most powerful and striking results in continuous probability. Intuitively, it says that if we perform a probabilistic experiment enough times and consider the distribution of the *average* of our results, then the distribution of the average will converge to look more like a Gaussian as our number of samples increases. More formally, we have the following.

Theorem 2. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. with mean μ and variance σ^2. Define*

$$A_n = \frac{X_1 + X_2 + \ldots + X_n}{n},$$

then

$$\frac{A_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty$$

In other words,

$$\mathbb{P}\left(\frac{A_n - \mu}{\sigma/\sqrt{n}} \leq \alpha\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx \text{ as } n \to \infty$$

Above, note that we have scaled and shifted our average to ensure that it has mean 0 and variance 1. To test your understanding, what type of Gaussian should our distribution converge to given that we instead examine the unscaled and unshifted $(X_1 + X_2 + \ldots + X_n)/n$?